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Adiabatical stability of density distributions under Vlasov dynamics*

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Abstract. It is shown using Vlasov dynamics that the density distribution corresponding to a mean field Bose condensate in an external time dependent potential is adiabatically stable whereas density distributions corresponding to finite temperature are not.

PACS. 05.20.Dd Kinetic theory – 65.90.+i Other topics in thermal properties of condensed matter

1 Introduction

Recently the interest in Bose-Einstein condensation has considerably increased, due to the fact that experimentally effects corresponding to the condensation phenomenon could be observed [1–3]. Theoretically the existence of Bose-Einstein condensation can be shown for free particles and on the basis of a mean field theory. *E.g.* in [4] a system corresponding to the experiment was considered. In [5] Michoel *et al.* treated a variation of a mean field theory, in [6] Kerson Huang considered the effect of an outside harmonic trap for a large but finite N, whereas in [7] the limit $N \to \infty$ was taken for temperature states in a scaling that mimics the scaling of large atoms. In this scaling the bosons have a large velocity that allows to consider them localized sharply in relation to the size of the trap potential. But if the potential is sufficiently deep an increasing part of the bosons will have negligible velocity and can be put into a ground state wave function of finite extension, whose kinetic energy due to the scaling becomes negligible.

The limit states obtained in this way, both ground states and temperature states, are of course time-invariant. But one can consider sequences of states with a similar scaling behaviour. This was done in [8]. Such states will not be time-invariant but with the appropriate choice of time scale the time evolution of these states obey the classical Vlasov dynamics where the density in phase space is the limit of Wigner distributions which in the chosen scaling become a positive measure on phase space.

Motivated by these results [7,8] we study in this paper the behaviour of the solutions of the Vlasov dynamics if we assume that the confining potential varies slowly in time. We should remember that Vlasov dynamics as a mean field theory preserves all L^p norms of the density distribution and therefore cannot explain entropy increase and hence convergence to equilibrium. If, on the other hand, we start with an equilibrium state, we can still hope that the system evolves in such a way that it remains close to equilibrium if we change the confining potential $V(0)$ to $V(T)$ but adiabatically slowly, *i.e.* at the time t $V(t)$ is given by $V(T\varepsilon)$ and the time interval runs from $0 \le t < T/\varepsilon$ and we consider the limit $\varepsilon \to 0$. Then $\rho(t)$ remains an equilibrium state up to order ε for $0 < t < T/\varepsilon$.

This turns out not to be true. We will obtain the following result: we consider only trap potentials that are harmonic. If we shift this trap potential adiabatically in space, then temperature states evolve to temperature states with fixed temperature and chemical potential (Sect. 3). If, on the other hand, we vary adiabatically the strength of the trap potential, then we are only able to show that adiabatically the Bose-Einstein condensate will remain a Bose-Einstein condensate (with some technical assumptions on the interaction potential). But (Sect. 4) for temperature states the invariance of the L^p norms leads to contradiction to the assumption that for adiabatic variation of the confining potential temperature states remain temperature states. The argument does not apply to the ground state since the density is a δ distribution in momentum space for which all L^p norms, $p > 1$, are infinite.

Confining potentials are global perturbations and we make no statement for local perturbations. The obtained result is in agreement to the observation that superfluids (that may be considered as Bose-Einstein condensates) react more promptly to changes of the environment, therefore global perturbations, *e.g.* can run through capillars.

2 The model

Motivated by the scaling behaviour of large atoms we consider the limit $N \to \infty$ of a system of N particles with

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corresponding Hamiltonian

$$H_N = N^{-2\gamma}\sum_i^N p_i^2 + \sum_i^N V(x_i) + \frac{1}{2N}\sum_{i\neq j}^N W(x_i - x_j). \tag{2.1}$$

with $\gamma = 1/\nu$, ν the dimension. Such a Hamiltonian arises naturally after the appropriate scaling for an atom with nuclear charge Z and N electrons

$$V(x_i) = \frac{1}{|x_i|}, \qquad W(x_i - x_j) = \frac{1}{|x_i - x_j|}$$

[9,10]. In [7] the corresponding system was considered for repulsive bosons with a trap potential $V = \lambda x^2$, as it would result from a homogeneous fermionic charge distribution. There it was shown that for sufficiently low temperatures Bose-Einstein condensation occurs. Similar bosonic systems with mean field Hamiltonian giving rise to Bose-Einstein condensation were considered in [6,11]. In addition to the evaluation of the energy it is also possible to make statements on the equilibrium state. With ω_N the equilibrium state for N particles and temperature β

$$\lim_{N\to\infty}\omega_N(e^{i\alpha N^{-\gamma}p_i + i\delta x_i}) = \mu(\alpha,\delta) \tag{2.2}$$

exists and $\mu(\alpha,\delta)$ is the characteristic function of a density distribution $\rho(x,p)$ that satisfies
for fermions:

$$\rho(x,p) = \frac{1}{1+\exp\left[-\beta(p^2 + V(x) + \int W(x-y)\rho(y,q)dydq) - \mu\right]},$$

for bosons provided $\rho(x,p) \geq 0$:

$$\rho(x,p) = \frac{1}{\exp\left[\beta(p^2 + V(x) + \int W(x-y)\rho(y,q)dydq) - \mu\right] - 1},$$

for classical systems with Boltzmann statistics:

$$\rho(x,p) = \exp[-\beta p^2]\exp\left[-\beta(V(x) + \int W(x-y)\rho(y,q)dydq) - \mu\right]. \tag{2.3}$$

For sufficently low temperatures it is necessary for bosons to put some particles into the condensate to satisfy $\rho(x,p) \geq 0$, *i.e.* a positive measure. Especially the groundstate corresponds to a density distribution which has the form

$$\rho_G(x,p) = \rho_G(x)\;\delta(p), \tag{2.4}$$

where $\rho_G(x)$ is determined by the condition

$$V(x) + \int W(x-y)\;\rho_G(y) = const\;\forall\; x$$

with *supp* $\rho_G(x) \neq 0$.

In general [7]

$$\rho(x,p) = \rho_{V_0}(x,p) + \lambda(x)\delta^\nu(p)$$

with

$$V_0(x) = \int \rho(x',p)W(x-x')dx'dp$$

$$\rho_{V_0}(x,p) = [\exp(\beta(p^2 + V(x) + V_0(x)) - 1]^{-1}$$

$$\lambda(x) \geq 0, \qquad supp\;\lambda = \{x : x^2 - \mu + V_0(x) = 0\}.$$

For $\beta = \infty$ $\lambda(x)$ reduces to $\rho_G(x)$ in (2.4).

We can consider sequences of states that scale in the same way (Eq. (2.2)) as the mean field models. The resulting states will in general be time dependent. The natural time scale is determined as the time scale in which particles with average momentum move for a finite distance, *i.e.* we consider

$$\lim_{N\to\infty}\omega_N\left(e^{-iH_N t N^\gamma}\,e^{i\alpha N^{-\gamma}p_i + i\delta x_i}\,e^{iH_N t N^\gamma}\right) = \mu(\alpha,\delta,t). \tag{2.5}$$

Provided the initial state satisfies that the expectation values of x_j^2 and p_j^2 are uniformly bounded independently of the particle number N, *i.e.*

$$\langle x_j^2\rangle_N \leq A, \qquad N^{-2\gamma}\langle p_j^2\rangle_N \leq B$$

and V and W are sufficiently regular it was proven in [8] that (2.5) exists and is the characteristic function of a density distribution that evolves according to the Vlasov equation, *i.e.* the mean field dynamics. It is given by

$$\frac{d\rho_t(x,p)}{dt} = \frac{\partial\rho(x,p)}{\partial x}p - \frac{\partial\rho(x,p)}{\partial p}\left\{\nabla V(x) + \int \nabla W(x-y)\rho_t(y,q)dydq\right\}.$$

The most suitable way to express this mean field dynamics reads:
$\rho_t(x,p)$ is given by the system

$$\frac{dx}{dt} = p$$
$$\frac{dp}{dt} = -\nabla V(x) - \int \nabla W(x-y)\rho_t(y,q)dydq \tag{2.6}$$

$$\rho_t(x,p) = \int \delta(x - x(y,q,t))\delta(p - p(y,q,t))\rho_0(y,q)dydq$$

where $x(y,q,t)$, $p(y,q,t)$ are the solutions with initial condition $x(y,q,0) = y$, $p(y,q,0) = \dot{x}(y,q,t=0) = q$.

This setting enables us to investigate whether the Bose-Einstein condensate (Eq. (2.4)) reacts more promptly than the rest to a change of the environment as we expect from a superfluid. We mimic this change of the environment by considering time dependent external potentials and check whether in the course of time the Bose-Einstein condensate remains a Bose-Einstein condensate. This is not possible for a sudden change of the potential. The initial state will not be a groundstate for the new potential and since the Vlasov dynamics preserves energy it cannot develop into a groundstate nor into a temperature state. Since the δ-type singularity in the momentum distribution of the ground state is preserved by (2.6) it cannot separate into a condensate and a smooth equilibrium distribution. If, on the other hand, we vary the external potential slowly then we can hope that in the adiabatic limit the Bose-Einstein condensate will remain a Bose-Einstein condensate.

We know that the adiabatic theorem holds in quantum mechanics [12], *i.e.* a groundstate remains a groundstate if $|\langle dV/dt\rangle| \ll E_1 - E_0$, the energy distance between the groundstate and the first excited state. For our Hamiltonian the groundstate of (2.1) is extended over a finite region, due to the repulsive interaction between the particles, therefore the kinetic energy per particle is of the order $N^{-2\gamma}$, in the first excited state one particle is excited in an area confined by supp $\rho_G(x)$, but otherwise in a constant potential, therefore $E_1 - E_0$ is also of the order $N^{-2\gamma}$, which means that H_N can only be allowed to change significantly over a time period of order $N^{2\gamma}$.

On the other hand, we can consider the Vlasov dynamics and compare the variation with the external potential with the time scale of the Vlasov dynamics $N^{-\gamma}$ (Eq. (2.5)). Thus slow with respect to Vlasov dynamics is fast in the sense of the previous scaling $N^{-2\gamma}$. Therefore we have to look for an adiabatic theorem on the basis of Vlasov dynamics. Since this is a classical evolution, we have to use the classical result [13]: If we stay in one dimension and the Hamiltonian H_t admits for all t action and angle variables then adiabatically the action variables remain constant.

A proof for an adiabatic theorem in general uses the fact that one can guess the appropriate time evolution and shows by perturbative arguments that the guess was correct. We will follow this method for the condensate. For temperature states the failure of this method does not imply instability of the equilibrium state, but we will be able to find contradictions to the assumption that the state remains a temperature state. Of course, Vlasov dynamics can only describe mean field effects and is therefore inadequate to treat typical quantum effects as interference.

3 Space translation of the potential

As a starting exercise we consider a shift of the external potential. We will observe that a shift of a harmonic external potential is sufficiently smooth so that no qualitatively different behaviour between Bose-Einstein condensate and temperature states can be observed.

Our Hamiltonian reads

$$H_N(t) = N^{-2\gamma}\sum_i^N p_i^2 + \sum_i^N \lambda(x_i - z(\varepsilon_N t))^2 + \frac{1}{2N}\sum_{i\neq j}^N W(x_i - x_j). \tag{3.1}$$

For the harmonic potential the Hamiltonian can be separated into center of mass coordinates and relative coordinates:

$$H_N(t) = N^{-2\gamma}P^2 + \lambda(X - \sqrt{N}\,z(\varepsilon_N t))^2 + N^{-2\gamma}\sum_{r=1}^{N-1} p_r^2 + \frac{1}{2N}\widetilde{W}(x_{r_1},\ldots,x_{r_{N-1}}) \tag{3.2}$$

with $P = \frac{1}{\sqrt{N}}\sum p_i$, $X = \frac{1}{\sqrt{N}}\sum x_i$.

The shift of the potential only effects the center of mass coordinates, the state corresponding to the relative coordinates will remain unchanged. For the center of mass the difference of the eigenvalues is $\sqrt{\lambda}\,N^{-\gamma}$, whereas $\langle\dot{V}\rangle = \sqrt{N}\,\dot{z}\langle x\rangle$, therefore ε_N has to be small compared to $N^{-(\gamma+1)/2}$. But this estimate is not optimal and can be improved by Vlasov dynamics. In fact, for the harmonic oscillator classical and quantum mechanical solutions coincide and the classical equation can be solved explicitly. We are interested in the variation of the macroscopic observable $\bar{x} = \lim \frac{1}{N}\sum x_i$ and in its time derivative with respect to the scaled time evolution, *i.e.* $\bar{p} = \lim N^{-\gamma}\frac{1}{N}\sum p_i$. If we introduce in the Vlasov equation the variables $x = \bar{x} + x_0$, $p = \bar{p} + p_0$, then we get with $\varepsilon_N = \varepsilon N^{-\gamma}$ (which coincides with the result of quantum mechanics only in one dimension, $\gamma = 1$)

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d^2\bar{x}}{dt^2} + \frac{d^2x_0}{dt^2}\\ &= -2\lambda(\bar{x} - z(\varepsilon t))\\ &\quad - 2\lambda x_0 - \int \nabla W(x_0 - y_0)\rho_0(y_0,q_0)dy_0dq_0.\end{aligned} \tag{3.3}$$

By assumption on $\rho_0(y_0, q_0)$

$$\frac{d^2x_0}{dt^2} = -2\lambda x_0 - \int \nabla W(x_0 - y_0)\rho_0(y_0,q_0)dy_0dq_0,$$

i.e. $x_0(y_0, q_0, t)$ is a solution corresponding to the time independent external harmonic potential. There remains

$$\frac{d^2\bar{x}}{dt^2} = -2\lambda(\bar{x} - z(\varepsilon t)).$$

Therefore $\bar{x}$ is a solution for the shifted harmonic potential with $\bar{x}(0) = \bar{p}(0) = 0$. $\bar{x} - z(\varepsilon t)$ is of order ε if $z(\varepsilon t)$ is twice differentiable for $t < T/\varepsilon$ now independent of the dimension $\nu = 1/\gamma$ [13]. If especially ρ_0 is a temperature distribution $\rho_\beta(x,p)$ of the various types (Eq. (2.3)) then $\rho_t(x,p) = \rho_0(\bar{x}+x_0, \bar{p}+p_0)$ remains close to a temperature distribution in the distributional sense, *i.e.*

$$\left|\int \rho(x,p)f(x,p)dxdp - \int \rho_\beta(x-\varepsilon t, p)f(x,p)dxdp\right| < \varepsilon c_f$$

for all t with $0 < t < T/\varepsilon$. Here f is a C^∞ function and c_f depends on f. It should be noted that the adiabatic theorem of quantum dynamics and the classical adiabatic theorem do not demand the same scaling of ε, because they do not ask the same question: even if the center of mass does not stay in the groundstate, the expectation value of x stays small as long as only the lower eigenstates contribute. But without a gap between the eigenvalues this cannot be controlled by the general theorem of quantum mechanics.

4 Variation of the strength

Now we consider the evolution of the ground state corresponding to

$$H_N = N^{-2\gamma} \sum_i^N p_i^2 + \lambda(\varepsilon_N t) \sum_i^N x_i^2 + \frac{1}{2N} \sum_{i \neq j}^N W(x_i - x_j). \tag{4.1}$$

Turning to center of mass coordinates and relative coordinates does not remove the time dependence for the relative coordinates anymore and therefore does not help in the sense that the Vlasov equation splits into an invariant part of the relative motion and a time dependent part of the center of mass motion. Instead we have to find a replacement for $z(\varepsilon t)$ in (3.3) as the movement $x_0(t, y)$ of the individual particles that is necessary to transform the initial ground state into the ground state corresponding to the changed confining potential. The velocity of this movement will be of order ε, determined by the velocity how quickly the strength of the confining potential is changed. The remaining task will be to show that the real movement of the particles will be a negligible oscillation around this movement x_0.

In order to find x_0 we first analyze the ground state as defined in (2.4) that for potentials that are radially symmetric satisfies

$$\int \nabla W(x-y)\rho_{Gt}(y)dy = \nabla V = 2\lambda(\varepsilon t)x, \qquad |x| \le a_t$$

$$\rho_{Gt}(x) = 0, \qquad |x| > a_t \tag{4.2}$$

with the normalization condition

$$\int_{|x|\le a_t} \rho_t(x)dx = 1.$$

We expect that the adiabatic evolution of a particle will oscillate around a path $x_0(t, y)$ determined by

$$\rho_{Gt}(x) = \int \delta(x - x_0(t, y))\rho_{G0}(y)dy. \tag{4.3}$$

Since we assumed a radial symmetric potential the resulting $x_0(t, y)$ will point to or from the center. We can obtain $x_0(t, y)$ by solving the equation for $p(x) \equiv \left.\frac{d}{dt}x_0(t, x)\right|_{t=0}$ and with $f(x)$ a smearing function

$$\begin{aligned}
&\int dx f(x)\frac{\partial\rho_{Gt}(x)}{\partial t} \\
&= -\iint dx f(x)\nabla\delta(x - x_0(t, y))\frac{dx_0(t, y)}{dt}\rho_{G0}(y)dy \\
&= \iint \nabla f(x)dx\delta(x - x_0(t, y))\frac{dx_0(t, y)}{dt}\rho_{G0}(y)dy \\
&= -\int f(x)dx\left(\frac{dp}{dx}\rho_{Gt}(x) + p\frac{\partial\rho_{Gt}(x)}{\partial x}\right).
\end{aligned} \tag{4.4}$$

This leads to the differential equation for $p_t(x)$

$$\frac{\partial\rho_{Gt}(x)}{\partial t} = -\frac{dp_t}{dx}\rho_{Gt}(x) - p_t\frac{\partial\rho_{Gt}(x)}{\partial x}$$

from which we can determine $p_t(x)$. For example we can consider in one dimension the potential $W(x-y) = |x-y|$. Then the corresponding groundstate is given by

$$\rho_t(x) = \frac{1}{a_t}, \qquad |x| \le a_t$$

$$2\lambda_t x = -\int_x^{a_t} \frac{1}{a_t}dy + \int_{-a_t}^x \frac{1}{a_t}dy = \frac{2x}{a_t}\cdot$$

Therefore

$$x_0(t, x) = \frac{a_t}{a_0}x = \frac{\lambda_0}{\lambda_t}x\,\cdot \tag{4.5}$$

Similarly the Coulombic interaction in three dimensions $W(x-y) = 1/(|x-y|)$ again leads to a constant density distribution $\rho_t(x) = c_t\Theta(a_t - |x|)$ [7]. (Remark: in [8] the validity of the Vlasov equation in the appropriate limit was only proven for non-singular potentials. On the other hand, the singularity is smoothened by the radial symmetric density distribution. Since the density remains radial symmetric and smooth, also for the Coulomb interaction the Vlasov equation describes correctly the behaviour.)

Now

$$\frac{1}{r}\,\frac{4\pi}{3}\,r^3 c_t = \lambda_t r^2$$

together with

$$c_t\,\frac{4\pi}{3}\,a_t^3 = \lambda_t a_t^3 = 1$$

so that again

$$x_0(t, x) = \frac{a_t}{a_0}\,x. \tag{4.6}$$

We have to notice that the ansatz (Eq. (4.3)) cannot be generalized to temperature states. First we do not know what $\rho_t(x, p)$ we have to choose, since we have to expect that temperature and chemical potential might change in time. In addition, for the groundstate we expect $x_0(t, x)$ to give approximately the correct answer. The particles have velocity of order ε, therefore the density in phase space $\rho_t(x)\delta(p - dx_0/dt)$ gives an expectation of the energy close to the groundstate energy up to order ε^2. For a temperature state particles will move even if the external potential is constant in time. This could be taken into account in (3.3) but a corresponding guess in the present case does not seem available.

We return to solve the Vlasov equation for the groundstate by the iteration ansatz and have to control that the deviation from x_0 defined *via* (4.4) remains an oscillation of $O(\varepsilon)$

$$\begin{aligned}
\frac{dx_n(t, x)}{dt} &= p_n(t, x) \\
\frac{dp_n(t, x)}{dt} &= -2\lambda(\varepsilon t)x_n(t, x) - \int \nabla W(x_{n-1}(t, x) \\
&\qquad - y_{n-1}(t, y))\rho_0(y)dy
\end{aligned} \tag{4.7}$$

starting with $x_0(t,x)$ given by (4.3). For x_0 we have $\dot{x}_0(0,x) \neq 0$, but for $n > 0$ we demand $\dot{x}_n(0) = 0$, since we start with the particle at rest. Lipschitz continuity guarantees that $\lim_{n\to\infty} x_n(t,x) = x(t,x)$ exists. Therefore we have only to control whether $|x_n(t,x) - x_0(t,x)| < c\varepsilon$, $|dx_n(t,x)/dt| < c\varepsilon$ for some $c > 0$ and $t \leq T/\varepsilon$ uniformly in n. With $\bar{x}_n(t,x) = x_n(t,x) - x_0(t,x)$ we get up to $O(\varepsilon^2)$

$$\begin{aligned}\frac{d^2\bar{x}_1(t,x)}{dt^2} &= -\frac{d^2x_0(t,x)}{dt^2} - 2\lambda(\varepsilon t)\bar{x}_1 - 2\lambda(\varepsilon t)x_0(t,x) \\ &\quad - \int \nabla W(x_0(t,x) - x_0(t,y))\rho_0(y)dy \\ &= -\frac{d^2x_0(t,x)}{dt^2} - 2\lambda(\varepsilon t)\bar{x}_1 \qquad (4.8)\end{aligned}$$

with initial conditions $\bar{x}_1(0) = 0$ and $d\bar{x}_1/dt = dx_0/dt = O(\varepsilon)$. Since we assume that $\lambda(\varepsilon t) \geq \lambda_0 > 0$ the solution of the homogeneous part of the differential equation with the above initial conditions is of the order ε. The solution of the inhomogeneous equation with trivial initial conditions is of order ε^2, if we assume $\lambda(t)$ to be three times differentiable. If we continue to estimate $\bar{x}_n$ we have to control

$$\int \nabla W(\bar{x}_{n-1}(t,x) - \bar{x}_{n-1}(t,y) + x_0(t,x) - x_0(t,y))\rho_0(y)dy. \qquad (4.9)$$

This will give a contribution of order ε and for controlling the differential equation we have to specify this contribution. $\bar{x}_{n-1}(t,y)$ is oscillating in t and varying in y. If as a first guess we assume that by integration over y the contribution will average out, we remain with

$$\frac{d^2\bar{x}_n}{dt^2} = -2\lambda(\varepsilon t)\bar{x}_n + 2\lambda(\varepsilon t)\bar{x}_{n-1} + O(\varepsilon^2)$$

and in the limit $n \to \infty$ the oscillation cannot control the term of order ε^2 for $t < T/\varepsilon$. Therefore it is necessary that (4.9) acts regulating and can be estimated by

$$-2\lambda(\varepsilon t)\bar{x}_n(t,x) + 2\lambda(\varepsilon t)\gamma(t)\bar{x}_{n-1}(t,x)$$

where $\gamma(t) < \gamma_0 < 1$, which means that $x_n(t,x)$ more or less oscillates in phase depending on x. Unfortunately we have not been able to show that this is really the case for all kind of particle interactions. We have only succeeded to control it for special interactions where we could use scaling arguments, though we believe that the result is more generally true.

Theorem
Let $\rho_t(x)$ be the groundstate density corresponding to an external potential $V = \lambda(\varepsilon t)x^2$ and an interaction $W(x-y) = v|x-y|^\mu$ where $v < 0$ for $0 < \mu < 2$ or $v > 0$ for $-\nu < \mu < 0$. Then $\forall\, T$ and all $\varepsilon > 0\ \exists\, c$ such that

$$|\rho(x,t) - \rho_t(x)| < c\varepsilon, \qquad t \leq T/\varepsilon \qquad (4.10)$$

where $\rho(x,q,t) = \rho(x,t)\delta\big(q - \dot{x}(x,t)\big)$ solves the Vlasov equation and $\dot{x}$ remains of order ε and $\rho(x,0) = \rho_0(x)$.

Proof
The normalization condition

$$\int \rho_t(x)dx = \rho_0$$

does not allow to make a scaling ansatz $\rho_t(x) = \alpha_t x^\gamma \Theta(c_t - x)$. But the scaling behaviour for the interaction allows the ansatz $x_0(t,x) = g_0(t)x$: we know that

$$v\int \nabla|x-y|^\mu \rho_t(y)d^\nu y = 2\lambda(t)x.$$

With (4.3) and the above ansatz we get

$$\begin{aligned}&v\int \nabla|x-y|^\mu \rho\left(\frac{y}{g_0(t)}\right) g_0^{-\nu}(t)d^\nu y \\ &= vg_0(t)^{\mu-1}\int \nabla|xg_0^{-1} - y|^{\mu-1}\rho_0(y)d^\nu y \\ &= g_0(t)^{\mu-1}2\lambda_0 x\cdot g_0^{-1} = 2\lambda(t)x\end{aligned}$$

and therefore with $g_0(0) = 1$ we get

$$g_0(t)^{\mu-2} = \frac{\lambda(t)}{\lambda(0)}. \qquad (4.11)$$

Induction allows to make the ansatz for (4.8)

$$\bar{x}_n(t,x) = \bar{g}_n(t)x$$

which leads to

$$\begin{aligned}\ddot{g}_0 + \ddot{\bar{g}}_n &= -2\lambda(t)(g_0 + \bar{g}_n) + (g_0 + \bar{g}_n)^{\mu-1}\frac{1}{x} \\ &\quad \times \int \nabla W(x-y)\rho_0(y)dy + O(\varepsilon^2).\end{aligned}$$

ρ_0 is of finite size, whereas we expect $\bar{g}_n$ to be $O(\varepsilon)$ so that we can expand

$$(g_0 + \bar{g}_n)^{\mu-1} = g_0^{\mu-1}\left(1 + (\mu-1)\frac{\bar{g}_n}{g_0}\right) + O(\varepsilon^2)$$

whereas $\int \nabla W(x-y)\rho_o(y)dy = 2\lambda_0 x$. Together with (4.11) this gives

$$\begin{aligned}\ddot{g}_0 + \ddot{\bar{g}}_n &= -2\lambda(t)(g_0 + \bar{g}_n) + (g_0 + \bar{g}_n)^{\mu-1}2\lambda(t) + O(\varepsilon^2) \\ &= -2\lambda(t)(2-\mu)\bar{g}_n + O(\varepsilon^2).\end{aligned}$$

With $\dot{\bar{g}}_n(0) = O(\varepsilon)$ we get $\bar{g}_n = O(\varepsilon)$ which proves (4.10).

5 Consequences for the temperature state

A path of the individual particle around which the real evolution will oscillate in the adiabatic limit is not available when we start with a density distribution corresponding to temperature. But this is not sufficient to conclude that for high temperature there does not exist an analogue of the adiabatic theorem.

But Vlasov dynamics is sufficiently informative to argue that indeed it does not hold: since $\rho(x,p,t) = \rho(x(t),p(t))$ it preserves all L^p norms in the course of time $0 \le p \le \infty$. This is so since $(x,p) \to (x(t),p(t))$ is a one parameter family (not group) of canonical transformations. Even if we take into account that temperature and chemical potential can be time dependent these two free parameters do not suffice to preserve the L^p norms. We will demonstrate this explicitly for Boltzmann statistics. Since for high temperatures Boltzmann statistics, Fermi statistics and Bose statistics approach one another, we have found a contradiction. First let us consider $W \equiv 0$. Then

$$\frac{dx}{dt} = p, \qquad \frac{dp}{dt} = -\lambda(\varepsilon t)x(t)$$

can be controlled by the results of [13] (p. 297). Then

$$\left|\frac{x^2(t)+p^2(t)}{\lambda(\varepsilon t)} - \frac{x^2(0)+p^2(0)}{\lambda(0)}\right| < c\varepsilon \quad \text{for} \quad t \le T/\varepsilon.$$

A state invariant under the initial time evolution is of the form

$$\rho(x,p) = f(p^2 + \lambda(0)x^2)$$

and will develop up to order ε to

$$f\left((p^2 + \lambda(\varepsilon t)x^2)\sqrt{\frac{\lambda(0)}{\lambda(\varepsilon t)}}\right). \tag{5.1}$$

If the initial state is a temperature state corresponding to $\beta(0)$ then the final state will be a temperature state with $\beta(t) = \beta(0)\sqrt{\lambda(0)/\lambda(\varepsilon t)}$ irrespective of the statistics (Eq. (2.3)).

Assume now $W \neq 0$ and that we start with a temperature state obeying Boltzmann statistics. Assume that this state evolves to a temperature with $\beta(t)$. Then, in dimension $\nu = 1/\gamma$

$$\begin{aligned}\rho(x,p,\beta_t) &= e^{-\beta_t p^2}\rho(x,\beta_t)\\ &= \exp[-\beta_t p^2]\exp\Big[-\beta_t(\lambda(\varepsilon t)x^2\\ &\quad + \int W(x-y)\rho(y,\beta_t)\beta_t^{-\gamma/2}dy\Big].\end{aligned} \tag{5.2}$$

Since $\rho_t(x,p)$ results from a flow in phase space

$$\int \rho_t(x,p)^q dxdp = \int \rho_0(x,p)^q dxdp \qquad \forall\, q$$

$\rho_t(x,p)$ can only be close to $\rho(x,p,\beta_t)$ for some β_t if for all q

$$\frac{\int \rho(x,\beta_t)^q dx}{\int \rho(x,\beta_0)^q dx} = \frac{\int e^{-\beta_0 q p^2}dp}{\int e^{-\beta_t q p^2}dx} = \sqrt{\frac{\beta_t}{\beta_0}}$$

which implies $\rho\,(x,\beta_t) = \rho\left(\sqrt{\beta_0/\beta_t}\,x,\beta_0\right)$. Due to the structure of $\rho(x,\beta_t)$ both terms in exp of (5.2) must have the same x-dependence or

$$\int W(x-y)e^{-\gamma y^2}dy = \alpha\, x^2$$

which cannot be satisfied for any interaction $\neq 0$.

We should finally remark that superfluidity is generally interpreted as a typical quantum effect. Here we have argued on the basis of pure classical dynamics. Quantum mechanics only occurs in the justification of the ground state distribution which turned out to be adiabatically invariant: in the above scaling it is the groundstate distribution for quantum bosons as well as for classical Boltzmann particles. But for quantum bosons it corresponds to a phase transition of second kind [7], therefore remains for sufficiently low temperatures, whereas for Boltzmann particles the δ-type singularity of the momentum immediately disappears for finite temperature.

6 Summary

We considered the behaviour of Vlasov dynamics under adiabatic variations of the confining potential. For arbitrary density distributions no general results are available except if the confining potential is only shifted in space and therefore only the center of mass motion is effected (Sect. 3). In general we found a counterargument that tells us that equilibrium distributions do not remain equilibrium distributions no matter how slowly the confining potential is changed (Sect. 5). But for the ground state distribution, *i.e.* the distribution with minimal energy (for bosons and Boltzmann statistics) the system remains in the ground state for slow variations.

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